Prophet Secretary Against the Online Optimal

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The Prophet Problem

$X_1 \sim \mathcal{D}_1$ $X_2 \sim \mathcal{D}_2$ $X_3 \sim \mathcal{D}_3$ $X_4 \sim \mathcal{D}_4$ $X_5 \sim \mathcal{D}_5$

- **Algorithm**: irrevocably select or discard at each step
- **Goal**: maximize the value selected
- **Benchmark**: all knowing Prophet: $\mathbb{E}[\max_i X_i]$

The Prophet Problem

35 $\sim \mathcal{D}_1$ **13** $\sim \mathcal{D}_2$ 42 $\sim \mathcal{D}_3$ 24 $\sim \mathcal{D}_4$ 17 $\sim \mathcal{D}_5$

Stop here!

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The Prophet *Secretary* Problem

Random permutation!

$X_4 \sim \mathcal{D}_4$ $X_5 \sim \mathcal{D}_5$ $X_1 \sim \mathcal{D}_1$ $X_2 \sim \mathcal{D}_2$ $X_3 \sim \mathcal{D}_3$

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Random permutation!

$\mathbf{24} \sim \mathcal{D}_4$ $X_5 \sim \mathcal{D}_5$ $X_1 \sim \mathcal{D}_1$ $X_2 \sim \mathcal{D}_2$ $X_3 \sim \mathcal{D}_3$

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The Prophet *Secretary* Problem

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The Prophet *Secretary* Problem

Random permutation!

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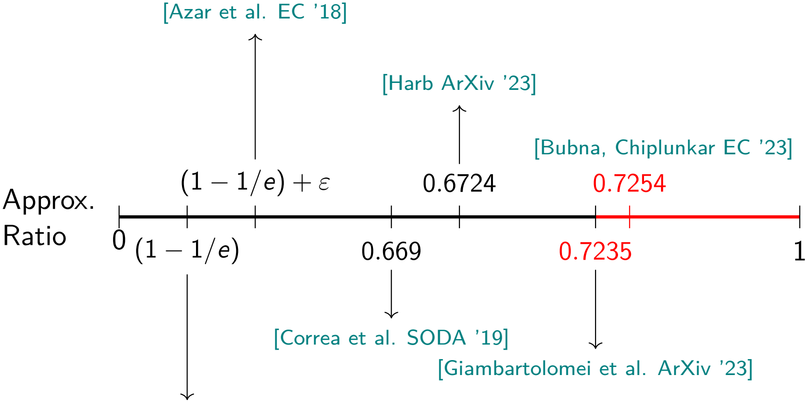
Stop here! Best value!

- **Algorithm**: irrevocably select or discard at each step
- **Goal**: maximize the value selected
- **Benchmark**: all knowing Prophet: $\mathbb{E}\left[\max_i X_i\right]$

State of the Art



[Esfandiari et al. ESA '15] prob formulation
[Correa et al. EC 2017]
[Ehsani et al. SODA '18]

Our Question

Our Benchmark: Online Optimal

Remaining variables

$$\text{OPT}(X|\mathcal{B}) = \mathbb{E}\left[\max(X, \frac{1}{|\mathcal{B}|}\sum_{X_i \in \mathcal{B}} \text{subproblem without } X_i)\right]$$

Current variable

Original benchmark: Prophet ($\mathbb{E}[\max_i X_i]$)

- Pessimistic benchmark!
- Many problems use Online Optimal (e.g. Matching [Papadimitriou et al. '21, Braverman et al. '21, Naor et al. '23], Prophet Inequalities [Niazadeh et al. '18], Pandora's Box-type problems [Weitzman '79, Chakraborty et al. '10, Fu et al '18, Singla & Segev '21, Liu et al. '21])

Our Results

Can we compete against the Online Optimal?
Yes!

PTAS

There exists an algorithm with

- $(1 - \varepsilon)$-approximation
- running time $n^{f(\varepsilon)}$

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QPTAS

There exists an algorithm with

- $(1-\varepsilon)$-approximation
- running time $n^{f(\varepsilon)\cdot \text{polylog } n}$

against the Online Optimal.

PTAS

There exists an algorithm with

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This presentation: First QPTAS, then PTAS

Online Optimal Revisited

DP for Online Optimal:

$$\mathrm{OPT}(X|\mathcal{B}) = \mathbb{E}\left[\max(X, \frac{1}{|\mathcal{B}|}\sum_{X_i \in \mathcal{B}} \text{subproblem without } X_i)\right]$$

Size is $n!$

Key observation: for constantly many different "types" (groups), DP size is polynomial

Online Optimal Revisited

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k_i: variables remaining in group i

$$\mathrm{OPT}(X | k_1, \ldots, k_g) =$$

$$\mathbb{E}\left[\max\left(X, \frac{1}{K}\sum_{i\in[g]} k_i \cdot (\text{subproblem with } k_i - 1)\right)\right]$$

Total variables remaining

Size is $\left(\frac{n}{g}\right)^g$ for g groups

QPTAS: $g = \text{const} \cdot \text{poly} \log n$
PTAS: $g = \text{const}$

Online Optimal Revisited

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Idea: discretize variables into g groups **while being close** to OPT

Warmup: Binary Distributions

All variables $X_i = \begin{cases} v_i & \text{, w.p. } p_i \\ 0 & \text{, w.p. } 1 - p_i \end{cases}$

Algorithm Steps:

Initial Instance $\mathcal{I}$

Warmup: Binary Distributions

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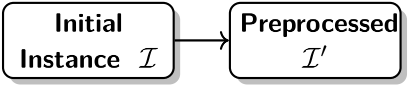
Algorithm Steps:



Preprocessing:
Remove "low" value X_i's

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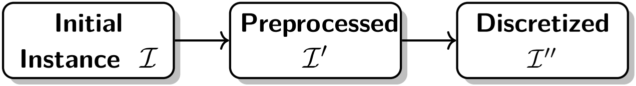

Preprocessing:
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Discretization:
round p_is & v_is

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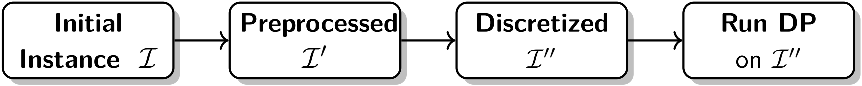


Preprocessing:
Remove "low" value X_i's

Discretization:
round p_is & v_is

Use thresholds of $\mathcal{I}''$ on $\mathcal{I}$ losing $O(\varepsilon)$

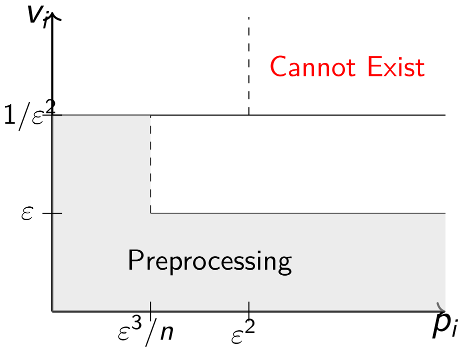
Preprocessing & Discretization

Preprocessing

- Scale optimal s.t. $\mathbb{E}[\max_i X_i] = 1$
- Remove all $v_i \leq \varepsilon$ or $p_i v_i \leq \varepsilon/n$

Total loss: $(1.5\varepsilon)^2$OPT.

Groups created: 0 (disregard X_is)



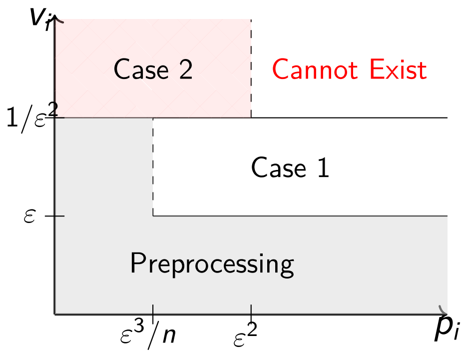
Preprocessing & Discretization

Discretization

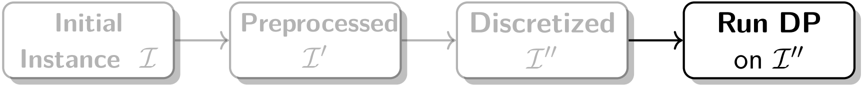
- **Case 1**: round down p_i and v_i to closest $(1+\varepsilon)^k$
- **Case 2**: set $v_i' = v_{\max}$ & set $p_i' \leftarrow \frac{v_i p_i}{v_{\max}}$

Total loss: ε^2OPT.

Groups created: $O\left(\frac{\log^2 n/\varepsilon}{\log^2(1+\varepsilon)}\right)$.



Final Step



Putting it all together:

- Loss is $O(\varepsilon)$OPT
- Total groups $O\left(\frac{\log^2 n/\varepsilon}{\log^2(1+\varepsilon)}\right) \Rightarrow$ runtime is $n^{\frac{\log^2 n/\varepsilon}{\log^2(1+\varepsilon)}}$

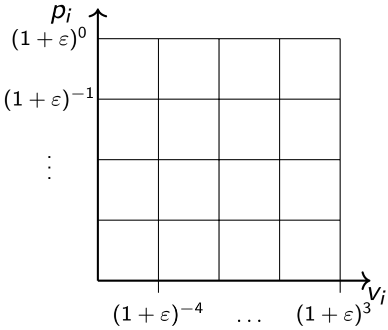
General distributions?

Generalizing to any distribution

- **Step 1: generalize to support size c (constant)**
- **Step 2: the general case**

Generalizing to any distribution

- **Step 1: generalize to support size c (constant)**
  - Variables of the form
    $$X_i = \begin{cases} v_1 & \text{w.p. } p_1 \\ v_2 & \text{w.p. } p_2 \\ \dots \\ v_c & \text{w.p. } p_c, \end{cases}$$
  - Variables in same group **if same support** after discretization
  - k: # of (value, probability) pairs
  - Total possible groups $\binom{c}{k} \approx k^c$
- Step 2: the general case



Generalizing to any distribution

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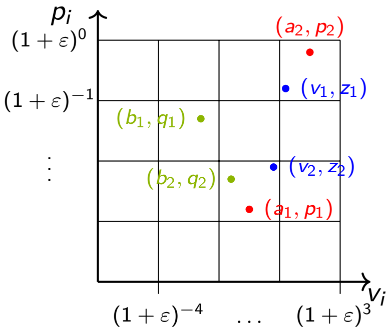


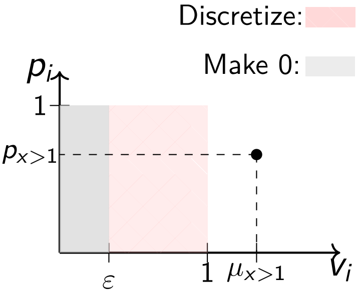
X_1 and X_2 **NOT** same group

X_1 and X_3 **IN** same group

Runtime: $O\left(n^{f(\varepsilon,c)\cdot \text{poly}\log n}\right)$

Generalizing to any distribution

- Step 1: generalize to support size c (constant)
- **Step 2: the general case**
  - Make 0 all $v_i \leq \varepsilon$
  - Collapse support above 1 to single point
  - Discretize support below 1
  - Reduce to constant support case with $c = O\left(\frac{\log 1/\varepsilon}{\log(1+\varepsilon)}\right)$



- $\mu_{x>1} = \mathbb{E}\left[X | X > 1\right]$
- $p_{x>1} = \mathbf{Pr}\left[X > 1\right]$

Runtime: $O\left(n^{f(\varepsilon,c)\cdot \text{poly} \log n}\right)$

PTAS

Problem with previous discretization:

$\#$ of probability values depends on n

Initial Instance $\mathcal{I}$

PTAS

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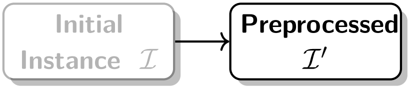


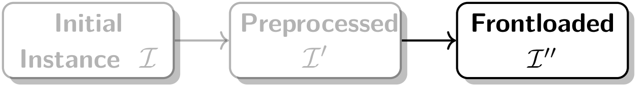
Preprocessing:

- Remove "low" value & low probability X_i's
- Discretize X_i's with "high"-er probability

PTAS

Problem with previous discretization:

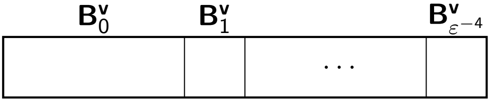
# of probability values depends on n



Frontloading:

- Split instance into blocks per support value $\mathbf{v}$
- At beginning of block $\mathbf{B}_i^{\mathbf{v}}$ flip coin to decide if accept $\mathbf{v}$

Support value $\mathbf{v}$



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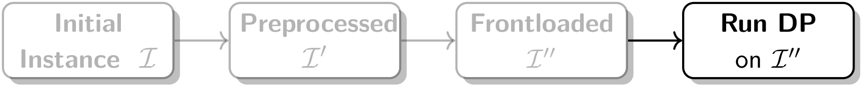

Use thresholds from $\mathcal{I}''$ on initial instance $\mathcal{I}$ losing $O(\varepsilon)$.

Runtime: $n^{(1/\varepsilon)^{\text{poly}1/\varepsilon}}$

Summary

Prophet Secretary against the **Online Optimal**

- **Result 1 (QPTAS):**
  $(1-\varepsilon)$ approximation in time $n^{f(\varepsilon)\cdot \text{polylog } n}$
- **Result 2 (PTAS):**
  $(1-\varepsilon)$ approximation in time $n^{f(\varepsilon)}$

Thank you!